

On-line list coloring of matroids

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ABSTRACT. A coloring of a matroid is *proper* if elements of the same color form an independent set. A theorem of Seymour asserts that a k -colorable matroid is also colorable from any lists of size k . In this note we generalize this theorem to the on-line setting. We prove that a coloring of a matroid from lists of size k is possible even if appearances of colors in the lists are recovered color by color by an adversary, while our job is to assign a color immediately after it is recovered. We also prove a more general weighted version of our result with lists of varying sizes. In consequence we get a simple necessary and sufficient condition for matroid list colorability in general case. The main tool we use is the multiple basis exchange property, which we give a simple proof.

1. Introduction

Let M be a loopless matroid on a ground set E . A coloring of the set E is *proper* if elements of the same color form an independent set of M . The *chromatic number* of M , denoted by $\chi(M)$, is the minimum number of colors needed to color properly the set E . In case of a graphic matroid $M = M(G)$, the number $\chi(M)$ is a well studied parameter known as the *arboricity* of the underlying graph G .

In [5] Seymour considered the following list coloring problem for matroids, in analogy to the list coloring of graphs. By a simple application of the matroid union theorem he proved that matroidal version of the choice number stays the same as the chromatic number.

THEOREM 1. (Seymour [5]) *Suppose that every element $e \in E$ of a matroid M is assigned a set of colors $L(e)$ of size at least $\chi(M)$. Then there is a proper coloring c of M satisfying $c(e) \in L(e)$ for each $e \in E$.*

M. Lasoń is supported by Polish National Science Centre under grant no N N206 568240.

W. Lubawski is supported by joint programme SSDNM.

In this paper we prove the *on-line* version of Seymour's theorem. Consider the following game played by Alice and Bob on a matroid M , in analogy to the graph coloring game introduced by Schauz [6] (cf. [8]). Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of colors, and let k be a fixed positive integer. In the first round Bob chooses arbitrary non-empty subset $B_1 \subseteq E$ and inserts color 1 to the lists of all elements of B_1 . Then Alice chooses some independent set $A_1 \subseteq B_1$ and colors its elements by color 1. In the second round Bob picks arbitrarily a non-empty subset $B_2 \subseteq E$ and inserts color 2 to the lists of all elements of B_2 . Then Alice chooses an independent subset $A_2 \subseteq B_2 \setminus A_1$ and colors its elements with color 2. And so on, until all lists will have exactly k elements. If at the end of the play the whole matroid is colored, then Alice is the winner. In the opposite case, Bob is the winner. Let $\tilde{\chi}(M)$ denote the minimum number k guaranteeing a win for Alice.

Our main result reads as follows.

THEOREM 2. *Every matroid M satisfies $\tilde{\chi}(M) = \chi(M)$.*

The proof relies on a multiple basis exchange property. Actually we prove a more general result, in which we allow for lists of varying sizes and coloring by sets of colors. This also gives the fractional version of the theorem (for fractional on-line list coloring of graphs see [1]).

2. The proof

We will need some notation. Let $\mathcal{P}(\mathbb{N})$ denote the family of all subsets of the set of positive integers \mathbb{N} (we use \mathbb{N} as the set of colors as well as the set of numbers). Let $\mathbf{w} : E \rightarrow \mathbb{N}$ be an assignment of *weights* to the elements of a matroid M . A \mathbf{w} -*coloring* of a matroid M is a function $W : E \rightarrow \mathcal{P}(\mathbb{N})$ such that every coloring c satisfying $c(e) \in W(e)$ is a proper coloring of M . Let $\ell : E \rightarrow \mathbb{N}$ be any function and let $L : E \rightarrow \mathcal{P}(\mathbb{N})$ be a list assignment of *size* ℓ , that is, for each $e \in E$ we have $|L(e)| = \ell(e)$. We say that M is \mathbf{w} -*colorable from lists* L if there is a \mathbf{w} -coloring W of M satisfying condition $W(e) \subseteq L(e)$ for each $e \in E$.

Now we may consider a generalized game on a matroid M with given functions \mathbf{w} and ℓ , which goes in the same way as described in the introduction, except that the goal of Alice is a \mathbf{w} -coloring of M from lists of size ℓ . If she has a winning strategy, then we say that M is *on-line* (\mathbf{w}, ℓ) -colorable.

Our aim is to prove a sufficient condition for the above property. We need two simple lemmas. The first is a well-known generalized exchange property. We will prove this lemma for the sake of completeness.

LEMMA 1. *Let I_1 and I_2 be independent sets of a matroid M . Then for every $X \subseteq I_1$ there exists $Y \subseteq I_2$ such that both sets, $(I_1 \setminus X) \cup Y$ and $(I_2 \setminus Y) \cup X$, are independent.*

PROOF. Let $I = I_1 \cap I_2$. We can restrict to the case where $I = \emptyset$. Indeed, if $I \neq \emptyset$, then consider matroid M with contracted set I and two independent sets $I_1 \setminus I$, and $I_2 \setminus I$. For $X \setminus I_2$ we get Y , which is also good in the previous case.

Now let $I_1 \cap I_2 = \emptyset$. Let M_1 be matroid M restricted to the set $X \cup I_2$, and let M_2 be matroid M restricted to the set $(I_1 \setminus X) \cup I_2$. Let $I_1 \cup I_2$ be their common ground set, and denote their rank functions by r_1, r_2 respectively. Observe that for each $A \subseteq I_1 \cup I_2$ we have:

$$\begin{aligned} r_1(A) + r_2(A) &= r(A \cap (X \cup I_2)) + r(A \cap ((I_1 \setminus X) \cup I_2)) \geq \\ &\geq r(A \cap (I_1 \cup I_2)) + r(A \cap I_2) \geq |A \cap I_1| + |A \cap I_2| = |A|, \end{aligned}$$

where the first inequality is just a submodularity of a rank function. From the matroid union theorem (see [4]) it follows that $I_1 \cup I_2$ can be covered by sets I'_1, I'_2 independent in M_1 and M_2 respectively, so also in M . Now $Y = I_2 \cap I'_2$ is a good choice, since $(I_1 \setminus X) \cup Y = I'_2$ and $(I_2 \setminus Y) \cup X = I'_1$. \square

As a corollary we get the multiple basis exchange property (see [2, 7]).

COROLLARY 1. (Multiple basis exchange property) *Let B_1 and B_2 be two bases of a matroid M . Then for every $X \subseteq B_1$ there exists $Y \subseteq B_2$, such that $(B_1 \setminus X) \cup Y$ and $(B_2 \setminus Y) \cup X$ are also bases.*

We say that a collection of sets I_1, \dots, I_k is a \mathbf{w} -cover of a set E if for each $e \in E$ we have $|\{i : e \in I_i\}| = \mathbf{w}(e)$. For a given subset $U \subseteq E$, let \mathbf{c}_U denote the characteristic function of U , that is, $\mathbf{c}_U(e) = 1$ if $e \in U$ and $\mathbf{c}_U(e) = 0$, otherwise. Now we prove the following inductive step lemma.

LEMMA 2. *Let I_1, \dots, I_k be a collection of independent sets in a matroid M forming a \mathbf{w} -cover of its ground set E . Then for every set $V \subseteq E$ there exists an independent set $I \subseteq V$ and independent sets I'_1, \dots, I'_k satisfying the following conditions.*

- (1) *The sets I'_1, \dots, I'_k form a $(\mathbf{w} - \mathbf{c}_I)$ -cover of E .*
- (2) *For each $e \in E$, if $e \in I'_s$ then $e \in I_t$ for some $t \geq s + \mathbf{c}_V(e)$.*

PROOF. Let $X_1 = (V \cap I_1) \setminus (I_1 \cap I_2)$. By Lemma 1 there exists $Y_2 \subseteq I_2$ such that $I'_1 = (I_1 \setminus X_1) \cup Y_2$ and $I''_2 = (I_2 \setminus Y_2) \cup X_1$ are independent. In general let $X_i = (V \cap I''_i) \setminus (I''_i \cap I_{i+1})$. So again by Lemma 1 there exists $Y_{i+1} \subseteq I_{i+1}$, such that $I'_i = (I''_i \setminus X_i) \cup Y_{i+1}$ and

$I''_{i+1} := (I_{i+1} \setminus Y_{i+1}) \cup X_i$ are independent. Let $I = X_k$. It is easy to see that conditions (1) and (2) are satisfied. \square

We are ready to prove the following generalization of the theorem of Seymour.

THEOREM 3. *Let ℓ be a given list-size function on the ground set E of a matroid M . If M is \mathbf{w} -colorable from lists of the form $L(e) = \{1, 2, \dots, \ell(e)\}$, $e \in E$, then M is on-line (\mathbf{w}, ℓ) -colorable.*

PROOF. We prove it by the induction on the number $\mathbf{w}(E) = \sum_{e \in E} \mathbf{w}(e)$. If $\mathbf{w}(E) = 0$, then \mathbf{w} is the zero vector and the assertion holds trivially. Suppose now that $\mathbf{w}(E) \geq 1$ and the assertion of the theorem holds for all \mathbf{w}' with $\mathbf{w}'(E) < \mathbf{w}(E)$. Let $V \subseteq E$ be the set of elements picked by Bob in the first round of the game. So, all elements of V have color 1 in their lists. Let I_1, \dots, I_k be a \mathbf{w} -coloring of M which exists by the assumption. By Lemma 2, there exist independent sets $I \subseteq V$ and I'_1, \dots, I'_k , such that I'_1, \dots, I'_k is a $(\mathbf{w} - \mathbf{c}_I)$ -cover of E . Now Alice colors all elements from I with color 1. By condition (2) of Lemma 2, matroid M is $(\mathbf{w} - \mathbf{c}_I)$ -colorable from lists $L'(e) = \{1, 2, \dots, \ell(e) - \mathbf{c}_V(e)\}$. The assertion of the theorem follows by induction. \square

Observe that the condition from the assumption of Theorem 3 is not only sufficient, but also a necessary for a matroid to be on-line (\mathbf{w}, ℓ) -colorable.

Taking $\mathbf{w} = (1, 1, \dots, 1)$ and $\ell = (k, k, \dots, k)$, with $k = \chi(M)$, we get immediately Theorem 2. Theorem 3 is an on-line generalization of Theorem 3 from [3]. Let us mention one of its off-line consequences.

COROLLARY 2. *If M is colorable from lists of the form $L(e) = \{1, 2, \dots, \ell(e)\}$, $e \in E$, then M is colorable from any lists of size ℓ .*

Acknowledgements

We would like to thank Jarek Grytczuk for many inspiring conversations, and additionally for the help in preparation of this manuscript.

References

- [1] G. Gutowski, Mr. Paint and Mrs. Correct go fractional, Electron. J. Comb. 18(1) (2011), RP 140.
- [2] J. Kung, Chapter 4 Basis-Exchange Properties, Theory of matroids, 62–75, Encyclopedia Math. Appl. 26, Cambridge Univ. Press, 1986.
- [3] M. Lasoń, The coloring game on matroids, arXiv:1211.2456.
- [4] J. Oxley, Matroid Theory, Oxford Univ. Press, 1992.

- [5] P. Seymour, A note on list arboricity, J. Combin. Theory Ser. B 72 (1998), 150-151.
- [6] U. Schauz, Mr. Paint and Mrs. Correct, Electron. J. Comb. 16(1) (2009), RP 77.
- [7] D.R. Woodall, An exchange theorem for bases of matroids, J. Combin. Theory Ser. B 16 (1974), 227-228.
- [8] X. Zhu, On-line list colouring of graphs, Electron. J. Comb. 16(1) (2009), RP 127.

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